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1974 J. Phys. A: Math. Nucl. Gen. 7 1094

(http://iopscience.iop.org/0301-0015/7/9/012)

J. Phys. A: Math., Nucl. Gen., Vol. 7, No. 9, 1974. Printed in Great Britain. © 1974

Field fluctuations in bistable gas lasers

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Received 15 October 1973

Abstract. A phenomenological gaussian noise source is introduced into the semi-classical field equation of a single-mode gas laser with a saturable absorber. The behaviour of the slowly varying amplitude is determined by studying the Fokker–Planck equation of the gaussian random process. The stationary intensity distribution shows two maxima caused by the bistability of the laser system. The noise source induces transitions between the two stable operating points. The approximate lifetimes of the states are evaluated, and are shown to be extremely long in the low noise situation. The introduction of the absorber cell broadens the linewidth of the lasing line and enhances intensity fluctuations. The phenomenological parameter characterizing the noise source is evaluated by comparison to quantum theory. The results derived for an ordinary laser are in agreement with quantum-mechanical calculations. The simple model allows one to consider arbitrary intensities in an inhomogeneously broadened single-mode gas laser system.

1. Introduction

An intra-cavity saturable absorber alters substantially the characteristics of a gas laser. Single-mode operation is preferred over wide ranges of the parameters (Beterov *et al* 1971, Lee *et al* 1968). The properties of the surviving mode differ considerably from those in an ordinary laser. One drastic example is the simultaneous occurrence of two stable operating points. In the bistable region the non-saturated absorption suffices to keep the total gain of the electromagnetic field below threshold over the whole bandwidth, but a large enough externally generated field is able to bleach the absorption so heavily that the laser begins to oscillate. Once the oscillating state is reached, it persists even if the external ignition field is switched off. The bistability of the laser operation gives rise to hysteresis phenomena (Lee *et al* 1968, Lisitsyn and Chebotaev 1968).

The above behaviour has been discussed theoretically in several papers (Kazantsev *et al* 1968, Greenstein 1972, Salomaa and Stenholm 1973). All these attack the problem within the framework of the semi-classical theory of gas lasers (Lamb 1964). Most of the steady-state characteristics can be calculated in this formulation, but, eg, it does not automatically describe the onset of oscillations. To follow the transient phenomena between the two operating points of a bistable gas laser, one must include external fields or intra-cavity noise sources. This paper concentrates on the effects of the latter ones. In addition to the improved description of the dynamics of the system, some new aspects of the steady-state behaviour can be extracted from this generalization, eg, a finite width of the lasing line (Lamb 1965).

The model of the laser consists of an electromagnetic field coupled to an atomic system. Both the atoms and the field are assumed to interact with independent heat baths causing fluctuations and relaxation effects in the systems considered (Lax 1966, Haken 1970). In the ordinary semi-classical theory the fluctuations are averaged out, and only the phenomenological decay terms remain. In this paper we shall retain the fluctuation force in the field equation, but neglect the random part in the matter equations. This simplification yields a numerically tractable model which is able to describe the transitions between the operating points of a bistable laser.

The fluctuations due to spontaneous emission are present in a theory which uses a fully quantized model for the system (Haken 1970, Kazantsev and Surdutovich 1969, Lax 1966, Scully and Lamb 1967). Technical noise sources (cavity vibrations, collisions, discharge effects, etc) must, however, be treated with some approximate methods. In this paper we adopt from the beginning a phenomenological model for the field fluctuations. The parameters describing the noise source must be determined either phenomenologically by a fitting procedure or by extracting them from the quantum-theoretical results. We prefer to use the semi-classical theory because for single-mode operation the quantum theory is already quite complicated at high intensities. On the other hand, in the high-intensity region we can expect good results from the semi-classical treatment by which we also obtain the exact solution (Stenholm and Lamb 1969).

Klimontovich *et al* (1972) have considered the fluctuations in ordinary gas laser systems with a generalized semi-classical model which takes into account both the field and polarization fluctuations. The extension of their results to the case of a laser with an intra-cavity absorber would be straightforward, but because of the mathematical complications we shall be content with a simpler model. Kazantsev and Surdutovich (1970) have considered the bistable laser quantum mechanically within nonlinear perturbation theory. Because the absorber is assumed to be heavily saturated, we cannot use their results, except near threshold, and secondly we discuss the inhomogeneously broadened case which is the most common situation in gas lasers. Qualitatively our results will show excellent agreement with theirs and this fact is taken to prove that our simplified model does not exclude any essential physical feature of the system.

We introduce a Langevin equation for the field and give the corresponding Fokker–Planck equation in § 2. The Fokker–Planck equation is reduced to two associated Langevin equations one of which depends only on the intensity of the field, and the other one describes the phase fluctuations. The steady-state intensity distribution is solved in § 3. In the same section we give an estimate for the linewidth of the oscillating mode. In § 4 we study some dynamical problems connected with a bistable laser system, and show that in the low noise limit the two operating points are metastable and have very long lifetimes. A brief discussion on the assumed single-mode stability is given in § 5.

2. Amplitude and phase equations

The cavity fluctuations due to thermal and technical noise sources are taken into account by the introduction of a random force $F(t)$ into the classical equation of motion (Salomaa and Stenholm 1973) for the slowly varying field amplitude

$$\frac{\mathrm{d}}{\mathrm{d}t}E = \frac{\Omega}{2}\left(\mathrm{i}\chi(E, E^*) - \frac{1}{Q}\right)E + F(t). \tag{1}$$

We prefer the use of a complex amplitude E to its separation into the modulus and phase variables. The laser is assumed to oscillate in a single mode. This case allows an exact semi-classical solution for the nonlinear susceptibility χ (Stenholm and Lamb 1969).

We deduce the stochastic properties of the complex fluctuating force $F(t)$ from the quantum-mechanical analogues. According to Haken (1970) (see also Lax 1966, Paul 1969, or Risken 1970) we can assume $F(t)$ to be approximately gaussian. The only non-vanishing correlation function is

$$\langle F(t)F^*(t')\rangle = K\delta(t-t'). \tag{2}$$

Since all the variables in equation (1) are c numbers it suffices to know the single intensi coefficient K. In contradistinction to quantum-mechanical calculations we obtain the same value for both $\langle FF^*\rangle$ and $\langle F^*F\rangle$. We also recall that the loss rate Ω/Q is related to the intensity coefficient K (see, eg, Haken 1970). Here we shall, however, regard K as an independent parameter. This is justified because we are mainly interested in the physical effects arising from the presence of the fluctuating force, and not in their exact numerical determination from first principles. One can also hope that part of the polarization fluctuations will be taken into account by the phenomenological parameter K. Klimontovich *et al* (1972) show that such a generalization makes K also depend on the field intensity. As this would again cause additional mathematical difficulties we will neglect it.

The Fokker–Planck equation (Lax 1966, Stratonovich 1967) corresponding to the Langevin equation (1) is

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial E}\left[\frac{\Omega}{2}\left(\mathrm{i}\chi-\frac{1}{Q}\right)EP\right]+\frac{K}{2}\frac{\partial^2}{\partial E\partial E^*}P+\mathrm{cc}, \tag{3}$$

where $P(E, E^*, t)$ is the probability of finding the values E and E^* at the time t. As we assumed a gaussian fluctuating force the equations (1) and (3) contain an equal amount of information.

Introducing the new dimensionless variables I and τ

$$E = \left(\frac{2\hbar^2\gamma_a\gamma_b I}{\wp^2}\right)^{1/2} \mathrm{e}^{\mathrm{i}\phi} \equiv (\beta I)^{1/2}\,\mathrm{e}^{\mathrm{i}\phi}, \tag{4}$$

$$t = \frac{Q\tau}{\Omega}, \tag{5}$$

we obtain from (3)

$$\frac{\partial P}{\partial \tau} = -\frac{\partial}{\partial I}\left(IG(I)P-ID\frac{\partial P}{\partial I}\right)-\frac{Q}{2}\mathrm{Re}(\chi(I))\frac{\partial P}{\partial \phi}+\frac{D}{4I}\frac{\partial^2 P}{\partial \phi^2}, \tag{6}$$

where

$$D = \frac{KQ\wp^2}{2\hbar^2\gamma_a\gamma_b\Omega} = \frac{KQ}{\beta\Omega}, \tag{7}$$

$$G(I) = -Q\mathrm{Im}(\chi(I))-1. \tag{8}$$

(We have adopted our notation from Salomaa and Stenholm 1973; γ_a and γ_b are the diagonal relaxation rates of the two-level atom and $\wp$ the dipole matrix element.)

The nonlinear susceptibility χ is in single-mode operation independent of the phase ϕ (see, eg, Stenholm 1971). This fact enables us to write down the Langevin equations

$$\frac{\mathrm{d}I}{\mathrm{d}\tau} = G(I)I + \frac{D}{2} + (2DI)^{1/2}\xi(\tau), \tag{9}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = \frac{Q}{2}\mathrm{Re}(\chi(I)) + \left(\frac{D}{2I}\right)^{1/2}\eta(\tau), \tag{10}$$

which are stochastically equivalent to the Fokker–Planck equation (6), if the new real fluctuation forces $\xi(\tau)$ and $\eta(\tau)$ have zero average values and satisfy:

$$\langle\xi(\tau)\xi(\tau')\rangle = \langle\eta(\tau)\eta(\tau')\rangle = \delta(\tau-\tau'), \qquad \langle\xi(\tau)\eta(\tau')\rangle = 0. \tag{11}$$

(The equivalency of (1) to (9) and (10) is due to the fact that both lead to the same Fokker–Planck equation, which is unique; see Stratonovich 1967).

The new Langevin equation (9) is independent of the phase ϕ, and hence we can solve the statistics of the intensity I by it or by the corresponding Fokker–Planck equation

$$\frac{\partial w}{\partial \tau} = -\frac{\partial}{\partial I}\left(IG(I)w - DI\frac{\partial w}{\partial I}\right), \tag{12}$$

where $w(I, \tau)$ is the probability distribution of the intensity I. Once we know the Green function of (12), all the expectation values of functions depending on I and τ can be evaluated.

In the phase equation (10), the intensity fluctuates slowly compared to the function $\eta(\tau)$. Introducing the phase variable

$$\Phi = \phi(\tau) - \phi(\tau') - \frac{Q}{2}\int_{\tau'}^{\tau} \mathrm{d}x\, \mathrm{Re}[\chi(I(x))], \tag{13}$$

we can write down the probability distribution of Φ for a fixed realization of $I(\tau)$:

$$\mathscr{G}(\phi, \tau; \phi', \tau') = \left(\pi D \int_{\tau'}^{\tau} \frac{\mathrm{d}x}{I(x)}\right)^{-1/2} \exp\left[-\Phi^2\left(D\int_{\tau'}^{\tau}\frac{\mathrm{d}x}{I(x)}\right)^{-1}\right] \tag{14}$$

(for a proof see Chandrasekhar 1943). We emphasize that the Green function (14) is random because of the randomness of $I(\tau)$. In principle the intensity fluctuations can be averaged out with the aid of the Green function of the Fokker–Planck equation (12). However, the evaluation of the integrals in (14) involves infinite order multi-time averages of $I(\tau)$ which makes the calculations extremely complicated.

In the following section we shall study the stationary intensity distribution and determine the linewidth of the field in the case when the intensity fluctuations can be neglected.

3. Steady-state characteristics

3.1. Intensity distribution

The true stationary distribution is obtained from (12) by requiring the probability current to vanish. Thus we have

$$I\left(D\frac{\partial w}{\partial I}-G(I)w\right) = 0 \tag{15}$$

which has the solution

$$w(I) = w_0 \exp\left(\frac{1}{D}\int G(I)\,\mathrm{d}I\right) \equiv w_0\,\mathrm{e}^{U(I)}. \tag{16}$$

The constant w_0 is fixed by the normalization of $w(I)$.

For a resonantly tuned mode we employ the expression

$$G(I) = \mathcal{N}(1+I)^{-1/2}-\mathcal{M}(1+\alpha I)^{-1/2}-1 \tag{17}$$

for the gain function (see, eg, Beterov *et al* 1971). This is the rate equation approximation (REA) result in the Doppler limit (ie, the Doppler width is much larger than the power broadened homogeneous width), and it can be shown to be accurate enough for our purposes (Salomaa and Stenholm 1973). The quantities $\mathcal{N}$ and $\mathcal{M}$ describe the pumping rates of the amplifier and absorber cell, respectively, and α is the relative saturability of the two active media. Here we always take $\alpha > 1$. The choice of a resonantly tuned situation is immaterial for the physics of the system. The exact gain function with arbitrary detuning and intensity can be evaluated, but because of the unnecessary additional mathematical complications we discuss (17) only. It can also be shown that the resonantly tuned laser exhibits all the qualitative features of interest here.

Inserting (17) into (16) we get

$$w(I) = w_0 \exp\left[\frac{2}{D}\left(\mathcal{N}(1+I)^{1/2}-\frac{\mathcal{M}}{\alpha}(1+\alpha I)^{1/2}-\frac{I}{2}\right)\right]. \tag{18}$$

Kazantsev and Surdutovich (1970) have derived a similar stationary intensity distribution for a homogeneously broadened laser in the limit $\alpha I \ll 1$.

The extreme values of (16) occur either at the zeros of $G(I)$ or at $I = 0$. These points correspond to the steady-state operating points predicted by ignoring the noise (cf equation (9) for $D = 0$). The type of operation of the laser system is classified according to the number of zeros of $G(I)$.

We have displayed the steady-state intensity distribution (18) in figure 1 for different values of $\mathcal{M}$ keeping $\mathcal{N}$ fixed. In the monostable region (the small-signal gain is above threshold, ie $G(0) > 0$) the gain function has a single zero, and accordingly $w(I)$ one maximum. Increasing the pumping current of the absorption cell, the laser is driven into the bistable region (in figure 1 $\mathcal{M} > 0{\cdot}5$). The small-signal gain is now below threshold but a large enough field can bleach the absorption and consequently the laser becomes able to oscillate. The gain function $G(I)$ thus has two zeros and also taking into account the point $I = 0$ we get three extreme values for $w(I)$. The two maxima correspond to the stable oscillating and non-oscillating solutions and the minimum to the unstable zero-gain point. A large enough absorption can completely prevent the laser from oscillating. In this case a single maximum occurs at $I = 0$, and $w(I)$ represents the amplified noise spectrum.

The moments of the intensity I can be calculated with the aid of the Laplace transform of $w(I)$:

$$\langle I^k\rangle = (-1)^k\left[\frac{\partial^k}{\partial s^k}\int_0^\infty \mathrm{d}I\,\mathrm{e}^{-sI}\,w(I)\right]_{s=0}. \tag{19}$$

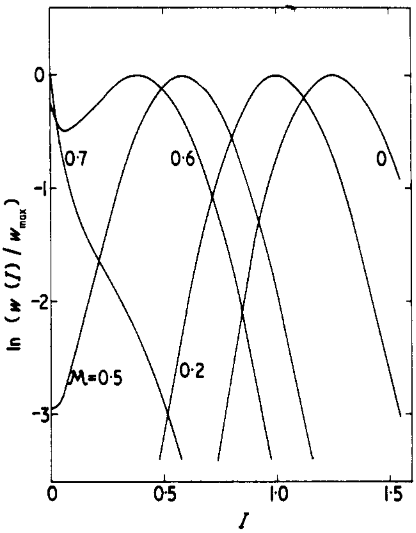

Figure 1. Semi-logarithmic plot of the stationary intensity distribution for various absorption cell pumping rates $\mathscr{M}$. The amplifier pumping rate $\mathscr{N} = 1{\cdot}5$, relative saturability $\alpha = 10$ and the diffusion coefficient $D = 0{\cdot}01$.

For a resonantly tuned ordinary laser ($\mathscr{M} = 0$) the Laplace transform of (18) yields

$$w(s) = w_0\, e^{\mathscr{N}/2D}\left[\frac{D}{1+Ds}+\frac{\mathscr{N}\sqrt{\pi D}}{(1+Ds)^{3/2}}\exp\left(\frac{(sD+1-\mathscr{N})^2}{D(sD+1)}\right)\operatorname{erfc}\left(\frac{sD+1-\mathscr{N}}{[D(sD+1)]^{1/2}}\right)\right]. \tag{20}$$

Evaluating the first two derivatives of (20) and exploiting the normalization condition $w(s=0) = 1$, we find

$$M \equiv \langle I\rangle = [D^2+\mathscr{N}D(1+\mathscr{N})+(\mathscr{N}^2-1+\tfrac{3}{2}D)A](D+A)^{-1}, \tag{21}$$

$$R \equiv \langle (I-M)^2\rangle = \{D^4-\mathscr{N}D^2[(\mathscr{N}+1)^2-\tfrac{1}{2}D(3+5\mathscr{N})]-AD[\mathscr{N}^3+\mathscr{N}^2(1-\tfrac{9}{2}D) -\mathscr{N}(1+\tfrac{1}{2}D)-\tfrac{11}{4}D^2+D-1]+A^2D(2\mathscr{N}^2+\tfrac{3}{2}D)\}(D+A)^{-2}, \tag{22}$$

where

$$A = \mathscr{N}\sqrt{\pi D}\exp\left(\frac{(\mathscr{N}-1)^2}{D}\right)\operatorname{erfc}\left(\frac{1-\mathscr{N}}{\sqrt{D}}\right). \tag{23}$$

In the limit $D\to 0$ the function A vanishes, if $\mathscr{N} < 1$, and approaches infinity if $\mathscr{N} > 1$. Carrying out the expansions of M and R to lowest order in D, we obtain

$$M = \begin{cases} D(1-\mathscr{N})^{-1} & \mathscr{N} < 1 \\ 2(D/\pi)^{1/2} & \mathscr{N} = 1 \\ \mathscr{N}^2-1+\tfrac{3}{2}D & \mathscr{N} > 1 \end{cases} \tag{24}$$

$$R = \begin{cases} D^2(1-\mathscr{N})^{-2} & \mathscr{N} < 1 \\ (2\pi-4)D/\pi & \mathscr{N} = 1 \\ 2\mathscr{N}^2 D & \mathscr{N} > 1. \end{cases} \tag{25}$$

We have illustrated the behaviour of M and R in figures 2 and 3.

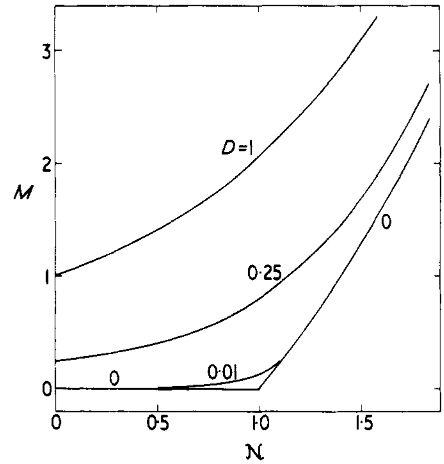

Figure 2. Average intensity of an ordinary resonantly tuned laser plotted against pumping of the cell at various noise levels.

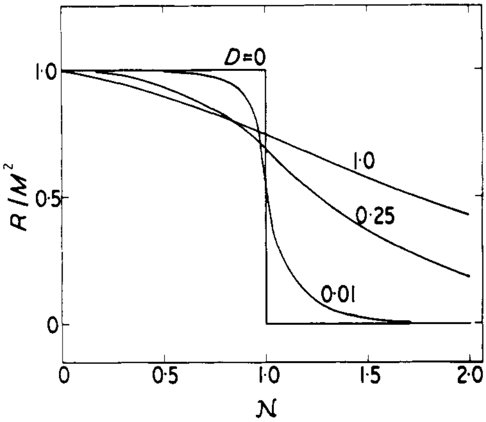

Figure 3. Relative variance corresponding to figure 2.

The comparison of the second central moment (22) to quantum-mechanical calculations enables us to give an estimate to D. Riska and Stenholm (1970) consider fluctuations arising from spontaneous emission, and derive the approximate result:

$$\left(\frac{R}{M^2}\right)_{\mathrm{QM}} = \frac{3\wp^2\Omega}{2\hbar\epsilon_0 V\gamma_a\gamma_b}\frac{2\mathcal{N}^2}{(\mathcal{N}^2-1)^2} \tag{26}$$

for a laser well above threshold ($\mathcal{N} > 1$). Evaluating the ratio R/M^2 by (24) and (25) and thereafter making it equal to (26), we find

$$D = \frac{3\wp^2\Omega}{2\hbar\epsilon_0 V\gamma_a\gamma_b} = \frac{3\hbar\Omega}{\epsilon_0 V\beta}. \tag{27}$$

With the values $\wp = 1{\cdot}6\times10^{-28}$ Cm, $V = 10^{-4}\,\mathrm{m}^3$, $\gamma_a = \gamma_b = 10^7$ Hz, $\Omega = 10^{15}$ Hz we obtain $D \sim 10^{-7}$.

The above order of magnitude estimate for D and equations (24)–(25) show that the fluctuations are important only very near threshold ($|\mathcal{N}-1| \sim D$). In the region well above threshold equation (24) reproduces the ordinary semi-classical steady-state intensity (see, eg, Stenholm 1971). The only difference is the additional noise contribution $\frac{3}{2}D$.

With a finite absorption ($\mathcal{M} > 0$) we have not succeeded in expressing the Laplace transform (19) in terms of elementary functions. A straightforward numerical integration also leads to difficulties since the exponent overflows easily for small values of D. In the limit $D \to 0$ we can, however, evaluate the moments of the intensity I by the method of steepest descent.

In the monostable region $w(I)$ has a single sharp maximum at $I = I_2$, in the vicinity of which we utilize the approximation

$$w(I) = w_0 \exp[a - \tfrac{1}{2}b(I-I_2)^2], \tag{28}$$

where according to (16)–(18)

$$a = U(I_2) = \frac{2}{D}\left(\mathcal{N}(1+I_2)^{1/2} - \frac{\mathcal{M}}{\alpha}(1+\alpha I_2)^{1/2} - \tfrac{1}{2}I_2\right), \tag{29}$$

$$b = -\frac{1}{D}\frac{\partial G}{\partial I}\bigg|_{I_2} = \frac{1}{2D}[\mathcal{N}(1+I_2)^{-3/2} - \alpha\mathcal{M}(1+\alpha I_2)^{-3/2}]. \tag{30}$$

The position of the maximum occurs at the zero of $G(I)$. The value of b must be positive which agrees with the ordinary stability criterion for the operating point (see, eg, Salomaa and Stenholm 1973).

For small values of D the most important contributions to the moments of the intensity come from the region where the approximate distribution (28) is valid. For the average intensity M and the variance R we find

$$M = I_2 + (2\pi b)^{-1/2}\exp(-\tfrac{1}{2}bI_2^2) \simeq I_2, \tag{31}$$

$$R = \frac{1}{b} - M(2\pi b)^{-1/2}\exp(-\tfrac{1}{2}bI_2^2) \simeq \frac{1}{b}. \tag{32}$$

When evaluating (31) and (32), we have performed the integration from $-\infty$ to $+\infty$ because the small tail of (28) below $I = 0$ is negligible.

In the bistable region we have to take into account also the maximum of $w(I)$ at $I = 0$. If the two peaks are clearly distinct we can write

$$w(I) \simeq w_0\{\exp(\zeta - \Gamma I) + \exp[a - \tfrac{1}{2}b(I-I_2)^2]\}, \tag{33}$$

$$\zeta = U(0) = \frac{2}{D}\left(\mathcal{N} - \frac{\mathcal{M}}{\alpha}\right), \tag{34}$$

$$\Gamma = -\frac{1}{D}G(0) = \frac{1}{D}(1 + \mathcal{M} - \mathcal{N}). \tag{35}$$

(note that the normalization constant w_0 is different in (28) and (33)). The validity of the approximation (33) requires:

$$\begin{aligned} &\zeta \gg a - \tfrac{1}{2}bI_2^2, \\ &\zeta - \Gamma I_2 \ll a, \end{aligned} \tag{36}$$

which two inequalities guarantee a negligible overlap of the two peaks.

A simple calculation yields for M and R in the bistable region the estimates

$$M = \left[\frac{1}{\Gamma^2}e^{\zeta} + I_2\left(\frac{2\pi}{b}\right)^{1/2} e^{a}\right]\left[\frac{1}{\Gamma}e^{\zeta} + \left(\frac{2\pi}{b}\right)^{1/2} e^{a}\right]^{-1}, \tag{37}$$

$$R = \left\{\frac{1}{\Gamma^4}e^{2\zeta} + \frac{2\pi}{b^2}e^{2a} + \frac{1}{\Gamma}\left(\frac{2\pi}{b}\right)^{1/2} e^{a+\zeta}\left[\left(I_2 - \frac{1}{\Gamma}\right)^2 + \frac{1}{b} + \frac{1}{\Gamma^2}\right]\right\}\left(\frac{1}{\Gamma}e^{\zeta} + \left(\frac{2\pi}{b}\right)^{1/2} e^{a}\right)^{-2} \tag{38}$$

(terms of the order $\exp(-\tfrac{1}{2}bI_2^2)$ have been dropped as in (31) and (32)).

In the non-oscillating region ($\mathcal{M}$ suffices to extinguish oscillations for all I) the stationary intensity distribution (18) is approximated by

$$w(I) \simeq w_0 \exp(\zeta - \Gamma I), \tag{39}$$

from which we obtain by (35)

$$M = D(1 + \mathcal{M} - \mathcal{N})^{-1}, \tag{40}$$

$$R = D^2(1 + \mathcal{M} - \mathcal{N})^{-2}. \tag{41}$$

We have plotted in figures 4 and 5 some numerically calculated curves for M and R in the region where the above approximate methods do not work accurately enough.

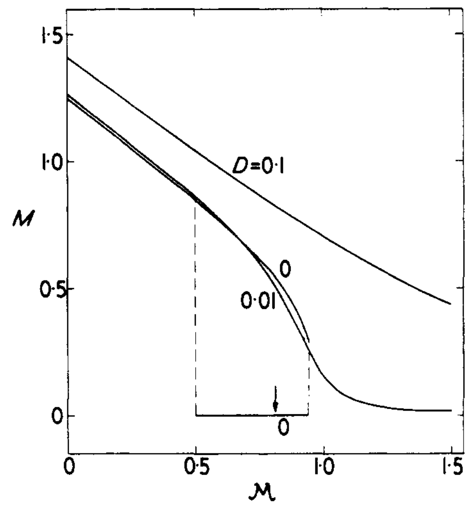

Figure 4. Average intensity of a laser with a saturable absorber plotted against absorption cell pumping rate $\mathcal{M}$ at various noise levels ($\mathcal{N} = 1{\cdot}5$ and $\alpha = 26{\cdot}0$). Bistable operation begins at $\mathcal{M} = 0{\cdot}5$ and oscillation is extinguished at the $\mathcal{M}$ indicated by the sudden drop in the curve $D = 0$. The arrow indicates the value at which the two maxima in the intensity distribution have equal height.

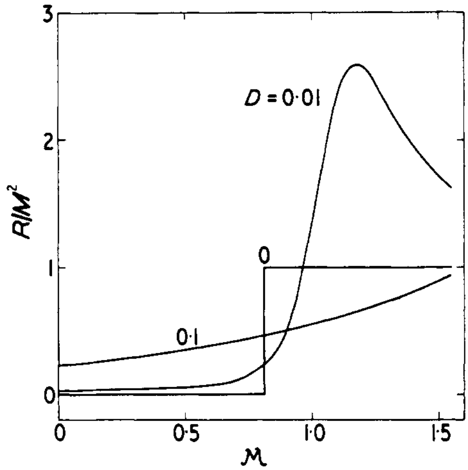

Figure 5. Relative variance corresponding to figure 4.

As a general feature we observe that the introduction of the absorber cell into the laser cavity tends, on the one hand, to diminish the average intensity and, on the other hand, broaden the intensity spectrum. The latter property is caused by two effects. One is that the spectrum may acquire two maxima and the other is that in the vicinity of the operating points fluctuations are enhanced more strongly when the absorber is present. From (30) and (32) we notice that the width of the intensity spectrum at the oscillating operating point is determined by the factor $-(G'(I_2))^{-1}$ which is smaller for an ordinary laser than for a laser with a saturable absorber.

Another interesting feature is that we always get a uniquely determined M also in the bistable region and even in the limit $D \to 0$. This stationary behaviour in the low noise limit is in contradiction with the results obtained by a theory which neglects noise. The latter predicts two stable operating points whereas from equation (37) we see that in the limit $D \to 0$ we obtain $M = 0$ for $a < \zeta$ and $M = I_2$ for $a > \zeta$. Going back to the expressions (29) and (34), and eliminating $\mathcal{N}$ by the equation $G(I) = 0$, we find that if $\mathcal{M}$, for a fixed intensity I of the oscillating operating point, exceeds the value

$$\mathcal{M}_{\mathrm{TR}} = \frac{\sqrt{1+\alpha I}}{2(\alpha-1)} \frac{(1+\sqrt{1+\alpha I})(\sqrt{1+\alpha I}+\sqrt{1+I})}{1+\sqrt{1+I}}, \tag{42}$$

the peak at $I = 0$ dominates so strongly that the system becomes monostable and has $M = 0$. In the opposite case ($\mathcal{M} < \mathcal{M}_{\mathrm{TR}}$) the oscillating operating point has such a large weight that we obtain $M = I$. Note that for all D the value $\mathcal{M}_{\mathrm{TR}}$ is the absorption at which the two maximum values of the stationary intensity distribution are the same.

The values $\mathcal{M}_{\mathrm{TR}}$ can be plotted in the $(\mathcal{N}, \mathcal{M})$ plane using I as a free parameter and we see that the curve (42) falls in the bistable region between the border lines of the monostable and non-oscillating regions (figure 6).

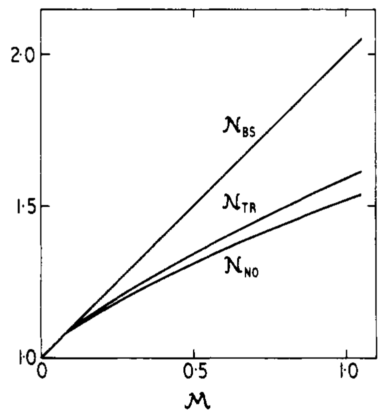

Figure 6. Type of operation of the laser system ($\alpha = 26$). For $\mathcal{N} > \mathcal{N}_{\mathrm{BS}}$ we obtain monostable behaviour and for $\mathcal{N} < \mathcal{N}_{\mathrm{NO}}$ the oscillating state disappears. At $\mathcal{N}_{\mathrm{TR}}$ the two maxima in the intensity distribution have equal height.

A comparison of (42) to the analytic expressions of the border lines (Salomaa and Stenholm 1973) yields

$$\frac{\mathcal{M}_{\mathrm{TR}}}{\mathcal{M}_{\mathrm{BS}}} = \tfrac{1}{2}(1+\sqrt{1+\alpha I}) \geqslant 1, \tag{43}$$

$$\frac{\mathcal{M}_{\mathrm{NO}}}{\mathcal{M}_{\mathrm{TR}}} = 2(1+\sqrt{1+I})\left[1+\left(\frac{1}{1+\alpha I}\right)^{1/2}\right]^{-1}\left[1+\left(\frac{1+I}{1+\alpha I}\right)^{1/2}\right] \geqslant 1 \tag{44}$$

(if $\mathcal{M} \leqslant \mathcal{M}_{\mathrm{BS}}$ the gain function $G(I)$ has a single zero and if $\mathcal{M} \geqslant \mathcal{M}_{\mathrm{NO}}$ the total gain is below threshold for all I).

The above result (the same feature is found by Kazantsev and Surdutovich 1970) that the bistability disappears in the true steady state does not, however, remove the hysteresis phenomena from the system. In § 4.1 we shall show that in the low-noise limit the transit times from one operating point to another become extremely long. This holds even for the case when the system is initially prepared to an operating point which has a negligible weight in the true stationary intensity distribution. Qualitatively it is easy to understand the long transit times with the aid of equation (9). For $I = 0$ and for small D the intensity grows slowly until it reaches a value comparable to the noise term $\frac{1}{2}D$. Then the negative G tends to damp the intensity and only a very rare realization of $\xi(\tau)$ is able to push the system above threshold. The two operating points are, therefore, metastable with such large lifetimes that the bistability is practically preserved.

3.2. *The linewidth of the laser*

In this section we calculate the linewidth of the field in a stationary state. If the system operates in the bistable region we assume that it stays at one of its metastable operating points, ignoring the possibility of a transition from one operating point to another.

The spectral density $S(\omega)$ of the slowly varying amplitude is determined by the formula (Stratonovich 1967):

$$S(\omega) = \frac{2\beta Q}{\Omega}\int_{-\infty}^{+\infty} \mathrm{d}\tau \exp\left(\frac{\mathrm{i}\omega Q\tau}{\Omega}\right)\langle\sqrt{I(\tau)I(0)}\exp[\mathrm{i}(\phi(\tau)-\phi(0))]\rangle, \tag{45}$$

where we have already introduced the dimensionless variables I and τ defined in (4) and (5). To calculate the expectation value in (45) we have to know the Green function $\mathscr{G}(I, \phi, \tau; I', \phi', 0)$ of the Fokker–Planck equation (6):

$$\langle\ \rangle = \int_0^\infty \mathrm{d}I \int_0^\infty \mathrm{d}I' \int_{-\infty}^{+\infty} \mathrm{d}\phi \int_{-\infty}^{+\infty} \mathrm{d}\phi' \sqrt{II'}\exp[\mathrm{i}(\phi-\phi')]\mathscr{G}(I, \phi, \tau; I', \phi', 0)P(I', \phi') \tag{46}$$

for $\tau > 0$ (for $\tau < 0$ we must interchange ϕ and ϕ'). $P(I', \phi')$ is the initial distribution. The phase ϕ is defined from $-\infty$ to $+\infty$ instead of the normal range from 0 to 2π by periodically continuing the latter one.

In the following we neglect the intensity fluctuations near the oscillating point I_2. This can be justified by linearizing the Langevin equation (9) at $I = I_2$, and noticing that the intensity fluctuations decay within a time $(-G'(I_2)I_2)^{-1}$ which is extremely short compared to the diffusion time $1/D$ in the equation (14), provided that $G'(I_2)$ does not vanish. Thus we can write

$$\mathscr{G}(I, \phi, \tau; I', \phi', 0) = \frac{w(I)}{(\pi D\tau/I_2)^{1/2}}\exp\left(\frac{(\phi-\phi'-\frac{1}{2}Q\tau\,\mathrm{Re}\,\chi(I_2))^2 I_2}{-D\tau}\right) \tag{47}$$

for times τ large compared to the correlation time of the intensity fluctuations. In deriving (47) we can replace $I(x)$ by I_2 in the integrals of the equation (14), because the corrections are of the order D.

For the initial distribution we take

$$P(I', \phi') = \delta(\phi')w(I'). \tag{48}$$

Since we neglect the possibility of transitions between the operating points and the small corrections arising from the replacement of I and I' in (46) by I_2 we can use for the intensity distribution $w(I)$ the approximate form $\delta(I-I_2)$. Inserting (47) and (48) into (46) and performing the Fourier transform we obtain

$$S(\omega) = 2\beta I_2 \frac{\Delta}{(\omega+\frac{1}{2}\Omega\,\mathrm{Re}\,\chi(I_2))^2+\Delta^2}, \tag{49}$$

where the linewidth of the lorentzian is

$$\Delta = \frac{\Omega D}{4QI_2}. \tag{50}$$

The shift of the centre frequency is due to the frequency-pulling effect caused by the dispersion of the optically active media.

The derived linewidth (50) agrees with Lamb (1965), if we use for D the expression

$$D = \frac{4(\bar{n}+\frac{1}{2})\hbar\Omega}{\epsilon_0 V\beta} \tag{51}$$

obtainable with the aid of the Callen–Welton theorem (see Klimontovich *et al* 1972). In (51) $\bar{n}$ is the average number of photons in the empty cavity and it can be ignored if $\hbar\Omega \gg kT$. Inserting (51) into (50), we obtain one half of the quantum-theoretical value (Scully and Lamb 1967). As we previously pointed out, the fluctuations are not entirely caused by thermal noise in the cavity. Spontaneous emission also contributes. Equation (27) predicts for D a value which is one and a half times the value given by the Callen–Welton theorem.

Because of this ambiguity of numerical factors we prefer to continue to keep D as a free adjustable parameter. Equation (50) offers one possibility to fix its value phenomenologically. Since the line of an oscillating laser is generally much narrower than the cavity linewidth Ω/Q we get an extremely small value of D when I_2 is of the order one.

This calculation shows that the bistable laser gives the same linewidth as an ordinary laser, except that the steady-state intensity has a diminished value. The essential approximation in the derivation is the assumed long lifetime of the oscillating state. This requires that the escape rate from the oscillating state is much smaller than the phase diffusion rate $D/4I_2$ (cf § 4.1).

4. Transient phenomena

4.1. Escape rates

This section is devoted to a semi-quantitative study on the temporal behaviour of a bistable laser. In order to describe the dynamics of the system, we must solve the time-dependent Fokker–Planck equation (6) with the desired initial conditions. If the system is prepared to the non-oscillating operating point, it is interesting to know how long it takes to reach the true stationary distribution. For this question it is sufficient to solve equation (12) only, and thus we are allowed to ignore phase fluctuations.

According to Stratonovich (1967), the complete solution of (12) is given by

$$w(I,\tau) = \sum_{n=0}^{\infty} e^{-E_n\tau} w_n(I) \int_0^{\infty} dI' w_n(I') \frac{w(I',0)}{w_0(I')}, \tag{52}$$

where the functions w_n are the eigensolutions of the equation

$$DIw_n'' + (D - IG)w_n' + (E_n - G - IG')w_n = 0 \tag{53}$$

with the eigenvalues E_n. The eigenfunction w_0 is the true stationary distribution (18) corresponding to the eigenvalue $E_0 = 0$. In our case $G(I)$ is a highly nonlinear function, which fact makes the solving of (53) extremely laborious.

The exact solution of the Fokker–Planck equation (12) is tractable only numerically for arbitrary pumping rates $\mathcal{N}$ and $\mathcal{M}$. To avoid the cumbersome computation process we try some approximate analytic methods. Extensive numerical calculations have been performed for a linearized gain function (see, eg, the review article by Lax and Zwanziger 1973 and references therein). In our case we should take into account also the quadratic terms. Kazantsev and Surdutovich (1970) have pointed out the similarity of the bistable laser to a brownian particle escaping over a potential barrier which has been discussed by Chandrasekhar (1943). In the following we derive the escape rates from the metastable operating points in a modified fashion and give conditions under which the results can be used.

We have solved the Green function of equation (12) in three different regions by linearizing the product $G(I)I$ (appendix 1). Near the origin, we obtain (A.9)

$$\mathscr{G}(I,\tau;0,0) = \frac{G_0}{D(1-e^{-G_0\tau})}\exp\left(-\frac{G_0 I}{D(1-e^{-G_0\tau})}\right), \tag{54}$$

where $G_0 = -G(0)$. This shows that local equilibrium is reached within a time G_0^{-1}. For small values of D, (54) describes accurately the behaviour of $w(I,\tau)$ in the region where the quantity

$$-I(G(I)-G(0))\mathscr{G}'-(G(I)-G(0)+IG'(I))\mathscr{G} \tag{55}$$

is negligible (see (12)).

By linearizing $G(I)I$ at the unstable operating point I_1 we get (A.14)

$$\mathscr{G}(I,\tau;I_0,0) = \left(\frac{2\pi DI_1}{g_1}(e^{2g_1\tau}-1)\right)^{-1/2}\exp\left[-[I-I_1-(I_0-I_1)e^{g_1\tau}]^2 \times\left(\frac{2DI_1}{g_1}(e^{2g_1\tau}-1)\right)^{-1}\right] \tag{56}$$

$$g_1 = I_1 G'(I_1). \tag{57}$$

The unstability follows from the positive value of g_1. If a realization $I(\tau)$ at time $\tau = 0$ takes the value I_0, it is extremely improbable to find it near I_1 in a region of width $2DI_1/g_1$ after a time $1/g_1$, unless

$$|I_0-I_1| \lesssim \left(\frac{2DI_1}{g_1}\right)^{1/2}. \tag{58}$$

We see that in the limit $D \to 0$ almost all realizations of $I(\tau)$ are reflected back towards the initial value $I(0)$.

The third approximate Green function (A.16) is given in the vicinity of the stable operating point I_2:

$$\mathscr{G}(I,\tau;I_0,0) = \left(\frac{2\pi DI_2}{g_2}(1-e^{-2g_2\tau})\right)^{-1/2}\exp\left(\frac{[I-I_2-(I_0-I_2)\,e^{-g_2\tau}]^2}{-(2DI_2/g_2)(1-e^{-2g_2\tau})}\right). \tag{59}$$

Formula (59) reveals the relevant decay rate $g_2 = -I_2G'(I_2)$. The approximate Green functions (54) and (59) show that local equilibrium is reached within times $1/G_0$ and $1/g_2$ (real time units are obtained by multiplying these by the cavity decay time Q/Ω).

Integrating (12) from 0 to I, we get

$$-j(I,\tau) = \frac{\partial}{\partial\tau}\int_0^I w(I,\tau)\,\mathrm{d}I = -G(I)Iw(I,\tau)+DI\frac{\partial w(I,\tau)}{\partial I}, \tag{60}$$

which is the rate at which the probability of finding a realization $I(\tau) \leqslant I$ decreases. The formal solution of (60) is given by

$$\left.\frac{w(I,\tau)}{w(I)}\right|_{I'}^{I''} = -\int_{I'}^{I''}\mathrm{d}I\,\frac{j(I,\tau)}{DIw(I)}. \tag{61}$$

For small values of D, only a negligible amount of probability can be accumulated in the vicinity of the unstable operating point $I = I_1$ (cf equation (56)), and therefore, $j(I,\tau)$ does not vary violently in that region. On the other hand, $(w(I))^{-1}$ is very sharply peaked at $I = I_1$ and the right-hand side of (61) can be evaluated by the method of steepest descent. We obtain:

$$\left.\frac{w(I,\tau)}{w(I)}\right|_{I'}^{I''} = -j(\tau)R \tag{62}$$

where

$$R = \frac{1}{2DI_1}\frac{1}{w(I_1)}\left(\frac{2\pi D}{G'(I_1)}\right)^{1/2}\left\{\mathrm{erf}\left[(I_1-I')\left(\frac{1}{D}G'(I_1)\right)^{1/2}\right]+\mathrm{erf}\left[(I''-I_1)\left(\frac{1}{D}G'(I_1)\right)^{1/2}\right]\right\}, \tag{63}$$

and the assumedly weak dependence on I has been omitted in $j(I,\tau)$.

The approximate solution of the Fokker–Planck equation (12) is given by a combination of (54), (59) and (62). After a time $1/G_0$ (or $1/g_2$) has passed, we can assume

$$\begin{aligned} w(I,\tau) &= C_1(\tau)w(I) \qquad I \leqslant I', \\ w(I,\tau) &= C_2(\tau)w(I) \qquad I \geqslant I''. \end{aligned} \tag{64}$$

The transfer of probability from one region to another in the initial transient from the δ function distribution to the local equilibrium is negligible if the stationary peaks are clearly distinct, and thus $C_1(0)$ and $C_2(0)$ can be determined to satisfy the initial preparation of the system. The solution (64) is taken to be valid for all τ which restricts our considerations to changes which take place slowly compared to $1/G_0$ or $1/g_2$. We choose I' and I'' in (62), (63) in such a manner that (64) approximately holds.

Inserting (64) into (62) we obtain

$$C_2-C_1 = -jR. \tag{65}$$

The total probability must be conserved, and because in the region (I', I'') only a negligible amount of probability can be accumulated, we get:

$$C_1P_1+C_2P_2 = 1, \tag{66}$$

$$P_1 = 1-P_2 = \int_0^{I_1} w(I)\,\mathrm{d}I. \tag{67}$$

The assumption that $j(I, \tau)$ is independent of I near I_1 implies that the probability escaping from the region $I \leqslant I'$ must arrive in the region $I \geqslant I''$, ie,

$$\dot{C}_1 P_1 = -\dot{C}_2 P_2 = -j. \tag{68}$$

The set of equations (65), (66) and (68) is easily solved and we find

$$C_1(\tau) = C_1(0)\,\mathrm{e}^{-\gamma_{02}\tau} + (1 - \mathrm{e}^{-\gamma_{02}\tau}), \tag{69}$$

$$C_2(\tau) = C_2(0)\,\mathrm{e}^{-\gamma_{02}\tau} + (1 - \mathrm{e}^{-\gamma_{02}\tau}), \tag{70}$$

where

$$\gamma_{02} = (P_1 P_2 R)^{-1}. \tag{71}$$

If the system is initially prepared to the non-oscillating operating point, we have $C_2(0) = 0$ and from (66) $C_1(0) = 1/P_1$. By (68), (69) and (71) we obtain

$$j_{0\to 2} \simeq \frac{1}{P_1 R}, \qquad \gamma_{02}\tau \ll 1. \tag{72}$$

For a system initially prepared at I_2, we find

$$j_{2\to 0} \simeq \frac{1}{P_2 R}, \qquad \gamma_{20}\tau \ll 1. \tag{73}$$

Because of the initial preparation of the system, equations (72) and (73) describe the rate at which realizations escape from one metastable state to the final state, since for times $\tau \ll \gamma_{02}^{-1}$ we can neglect the backflow.

For small values of D the error functions in (63) both equal approximately one. Calculating the weights P_1 and P_2 by the method of steepest descent (cf § 4.1) we obtain from (63), (72) and (73)

$$j_{0\to 2} = -I_1 G(0)\left(\frac{G'(I_1)}{2\pi D}\right)^{1/2} \exp(U(I_1) - U(0)), \tag{74}$$

$$j_{2\to 0} = \frac{I_1}{2\pi}(-G'(I_2)G'(I_1))^{1/2} \exp(U(I_1) - U(I_2)). \tag{75}$$

In the low-noise limit, $D \to 0$, the escape rates behave as $\exp(-\epsilon^2/D)$ since $U(I_1) < U(0)$, $U(I_2)$. Their vanishing implies that the system stays infinitely at the point where it is initially prepared. This proves the statement that in the low-noise limit the operating points are metastable. The results (74) and (75) are illustrated in figure 7.

The formula (61) is exact. The essential approximations are done in (64). The assumption of local equilibrium requires that the net flow out of a stable region in a characteristic time, needed to achieve local stationarity, should be small compared to the net probability in that region. For example, if the system is initially prepared to the point $I = 0$, the following inequality must be satisfied

$$\frac{1}{G_0} j_{0\to 2} \ll 1. \tag{76}$$

Inserting (74) into (76) and assuming the linearized form of $G(I)$ to be valid for $I < I_1$, we find

$$I_1\left(\frac{G'(I_1)}{2\pi D}\right)^{1/2} \exp\left(\frac{-G_0 I_1 + \frac{1}{2}G'(I_1)I_1^2}{D}\right) \ll 1, \tag{77}$$

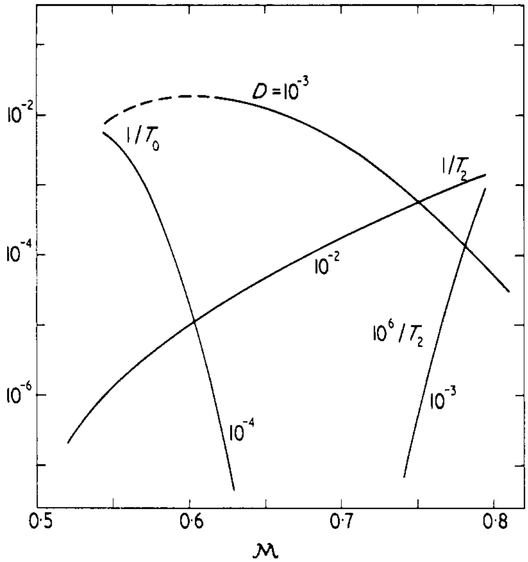

Figure 7. Inverse lifetimes in units of Ω/Q of the metastable states of a bistable laser plotted against the absorption cell pumping rate $\mathcal{M}$ ($\mathcal{N} = 1{\cdot}5$ and $\alpha = 26$). For the applicability of the drawn curves see the text (the left-hand side of (80) is $\sim 10^{-3}$ at $\mathcal{M} = 0{\cdot}6$).

where

$$-G_0 + G'(I_1)I_1 = 0. \tag{78}$$

Utilizing (78) to eliminate I_1, and determining the point at which the left-hand side of (77) acquires its maximum, we get the condition

$$\frac{G_0^2}{G'(I_1)} > D\sqrt{2}. \tag{79}$$

Neglecting the difference between $G'(I_1)$ and $G'(0)$, we obtain from (17) and (79)

$$\frac{\sqrt{2}(1+\mathcal{M}-\mathcal{N})^2}{\alpha\mathcal{M}-\mathcal{N}} > D. \tag{80}$$

If the small-signal gain is very near threshold, and the relative saturability $\alpha \gg 1$, this condition gives a small upper bound to D, indicating that noise can relatively easily set on the oscillations. In a similar manner one can show that the condition

$$\frac{1}{g_2} j_{2\to 0} \ll 1 \tag{81}$$

must be satisfied when the system is initially at I_2. Inserting (75) and the value $g_2 = -I_2 G'(I_2)$ into (81), we find

$$\frac{1}{2\pi}\frac{I_1}{I_2}\left(-\frac{G'(I_1)}{G'(I_2)}\right)^{1/2} \exp\left(-\frac{1}{D}\int_{I_1}^{I_2} G(I)\,\mathrm{d}I\right) \ll 1, \tag{82}$$

where the $U(I)$ functions have been expressed with the aid of (16). In the limit $I_1 \to I_2$, $G(I)$ can be approximated by a parabola, and we see that (82) holds for

$$\tfrac{2}{3}(I_2 - I_1)G_{\max} > D. \tag{83}$$

If the absorption is large enough the laser operates near the extinction region ($I_1 = I_2$) and small fluctuations can shut off the oscillation.

To clarify the physical meaning of the above considerations we assume that the system is initially prepared near $I = 0$ and neglect all the realizations of $I(\tau)$ which have a larger value than I_1. Setting $C_2 \equiv 0$, we obtain from (65) and (68)

$$j = \frac{1}{P_1 R} \exp\left(-\frac{\tau}{P_1 R}\right) \tag{84}$$

(note that because all the escaped realizations are neglected we do not have to satisfy (66)). At time τ $j\mathrm{d}\tau$ realizations escape. The average lifetime of the metastable state $I = 0$ is thus

$$T_0 = \int_0^\infty \mathrm{d}\tau \, j(\tau)\tau = P_1 R. \tag{85}$$

In a similar manner one can show that the lifetime of the metastable state I_2 equals

$$T_2 = P_2 R. \tag{86}$$

The possibility of escaping from the oscillating operating point increases the width of the lasing line. This is approximately taken into account by multiplying the Green function (47) by a factor $\exp(-\tau/T_2)$ (cf equation (84)). Redoing the calculations of § 3.2, we find for the linewidth

$$\Delta = \frac{\Omega D}{4QI_2} + \frac{\Omega}{QT_2}. \tag{87}$$

In the low-noise limit $1/T_2$ behaves as $\exp(-\epsilon^2/D)$, and the last term in (87) can be neglected in comparison to the phase diffusion term.

4.2. Dynamics of the expectation values

Another way to investigate the dynamical behaviour of the system is to study the equations of motion oi the expectation values. Multiplying (12) by an arbitrary function $f(I)$ and integrating by parts we find

$$\frac{\partial}{\partial \tau}\langle f \rangle = D\langle If'' + f' \rangle + \langle IGf' \rangle + \Big|_0^\infty I[f(D-G)w' - f'Dw]. \tag{88}$$

In the following we consider only such functions for which the last term vanishes.

For the average value $M = \langle I \rangle$ and the variance $R = \langle I^2 \rangle - M^2$ we obtain from (88) the equations

$$\frac{\partial}{\partial \tau} M = D + \langle IG \rangle, \tag{89}$$

$$\frac{\partial}{\partial \tau} R = 2DM + 2\langle I^2 G \rangle - 2M\langle IG \rangle. \tag{90}$$

The distribution $w(I, \tau)$ is completely characterized by its moments. Writing the equations of motion for all the moments (or combinations of these, eg, cumulants or central moments) and expressing the expectation values $\langle I^k G(I) \rangle$ in terms of these we obtain an infinite set of coupled equations which determine the temporal behaviour of the system.

In the following we assume R to be small and the higher central moments to vanish. Expanding G in powers of $(I-M)$ we find from (89) and (90) a closed set of equations

$$\frac{\partial}{\partial\tau}M = D+MG(M)+R(G'(M)+\tfrac{1}{2}MG''(M)), \tag{91}$$

$$\frac{\partial}{\partial\tau}R = 2DM+2R(G(M)+MG'(M)). \tag{92}$$

If the system is initially prepared so that $w(I,0) = \delta(I-M_0)$, we see that for small values of D the distribution moves rapidly to one of the operating points, provided that

$$D \ll |M_0G(M_0)|. \tag{93}$$

During this transient the distribution broadens only slightly. This supports the assumptions made in § 4.1 on local equilibrium.

We consider only the situation where the system is initially prepared to $I = 0$. As long as the distribution is confined to $I \lesssim I_1$ we can approximate

$$G(I) = -G_0+G_1I. \tag{94}$$

Inserting this into (91) and (92), we obtain the steady-state solutions

$$M = 2D[G_0+(G_0^2-8DG_1)^{1/2}]^{-1}, \tag{95}$$

$$R = 4D^2[G_0+(G_0^2-8DG_1)^{1/2}]^{-2}. \tag{96}$$

(In fact we get three solutions but one of these has negative R and one can be excluded by stability considerations.) From (95) we see that M becomes imaginary for

$$D > \frac{G_0^2}{8G_1} = \frac{(1+\mathscr{M}-\mathscr{N})^2}{4(\alpha\mathscr{M}-\mathscr{N})}. \tag{97}$$

As the linearization (94) still can be expected to be relatively good the result evidently indicates that noise is able to remove the system to the oscillating operating point. Comparison of (97) to (80) further supports this argument. If we neglect R totally in (91) we find that $\partial M/\partial\tau$ is positive in the linearized region (94) if D exceeds twice the value given in (97).

We do not integrate the equations (91) or (92) numerically in this paper. However, we point out their usefulness in evaluating steady-state properties. For example, we see from (92) that for the system at $M \simeq I_2$, the steady-state variance R is given by

$$R = -\frac{D}{G'(M)}. \tag{98}$$

Near the extinction border ($G'(M) = 0$) the fluctuations are enhanced greatly. Similarly one sees that for a system, prepared to $I = 0$, we have from (92)

$$R = \frac{D}{G_0}. \tag{99}$$

The two metastable operating points have a different nature. The non-oscillating region is bound both from below and above ($0 \leqslant I < I_1$) whereas the oscillating operating point has only a lower limit ($I > I_1$). This fact tends to make the oscillating state more stable against fluctuations, if $G'(I_2)$ and G_0 are of the same order (cf figure 7), and is clearly demonstrated if we calculate the steady state M and R by (90) and (91) for a linearized gain function

$$G = -G'(I_2)(M-I_2). \tag{100}$$

The steady-state average intensity turns out to be I_2 and the variance is given by (98) for $M = I_2$. The linearized gain function (100) always yields a stable solution. Instabilities can be expected only in the region where the linearization (100) breaks down.

5. Discussion

We have introduced a phenomenological gaussian random force into the equation of motion of the slowly varying mode amplitude. The results obtained for an ordinary laser are in satisfactory agreement with calculations based on fully quantized models. Kazantsev and Surdutovich (1970) present a Fokker–Planck equation equivalent to equation (6), except that we have ignored the intensity dependence of the diffusion coefficient D. This simplification does not exclude any physical feature of the system, and by extracting D from experimental data we are able to take into account all noise sources in an approximate way. In the low-noise limit a weak intensity dependence could be included in D by letting it have a different value at the oscillating and non-oscillating operating points. For small values of D the two metastable states have long enough lifetimes to allow experimental determination of stationary operating characteristics. To a first approximation the technical noise sources do not depend on whether the bistable laser oscillates or not, because the pumping currents and the pressures of the two cells are kept constant. Measuring the value of D at the oscillating and non-oscillating operating points yields information on the saturation properties of the quantum-mechanical noise sources (cf Klimontovich *et al* 1972, Kazantsev and Surdutovich 1969).

In this paper the noise source is assumed to be gaussian. This requires that a field fluctuation and its polarization response have a much shorter correlation time than the intensity correlation time. Such a situation occurs for $\gamma Q/\Omega \gg 1$ (see, eg, Haken 1970). The high relative saturability of the two cells is obtained, if we have

$$\alpha = \left(\frac{\wp_{\text{abs}}\gamma_{\text{amp}}}{\wp_{\text{amp}}\gamma_{\text{abs}}}\right)^2 > 1. \tag{101}$$

In case the dipole matrix elements are equal, eg, in a He–Ne amplifier and Ne absorber, equation (101) is satisfied for $\gamma_{\text{amp}} > \gamma_{\text{abs}}$, which holds if the absorber cell is at a lower pressure than the amplifier cell (see, eg, Lee *et al* 1968). The homogeneous width γ_{abs} may turn out to be of the same order as the cavity linewidth Ω/Q, but the restriction $\gamma Q/\Omega \gg 1$ can be shown to be too strong (Lax and Zwanziger 1973).

We have considered single-mode operation and neglected the radial dependence of the field. The higher transverse modes are excluded because of their assumedly small Q values (to satisfy this some special arrangements in the experimental set up may be needed). However, one must remember that the higher transverse modes may play a role in the dynamical behaviour of the system (Lax *et al* 1972). Focusing (or defocusing) of the beam can be approximately included in the parameter α, but replacing I in $G(I)$ by a radial average may introduce appreciable error when I is large (Beterov *et al* 1971). For small gas densities self-focusing of the field is not of importance, and the radial average must be formed only when investigating the temporal development of the mode.

According to Salomaa and Stenholm (1973) the fact that non-saturated absorption is able to keep most cavity modes below threshold, is primarily responsible for the preferred single-mode operation. At the oscillating mode the absorption is heavily bleached away causing minor power losses only. The stability of the single-mode operation is investigated by evaluating the small-signal gain functions of the non-oscillating modes. Including the noise source, we find for a weak mode, I_2 say, from (9)

$$\frac{dI_2}{d\tau} = G_2(I_1, 0)I_2 + \tfrac{1}{2}D + \sqrt{2DI_2}\xi_2(\tau), \tag{102}$$

where the small-signal gain $G_2(I_1, I_2 = 0)$ is given by Salomaa and Stenholm (1973), and because the two modes have different eigenfrequencies the fluctuation force $\xi_2(\tau)$ is uncorrelated with the fluctuation force of mode 1. If $G_2 < 0$ for some value of I_1, the state $(I_1, I_2 \simeq 0)$ is metastable and the larger $-G_2/D$ is the longer the lifetime (cf § 4.1). We must also remember that I_1 is a random variable, but since its fluctuations are slow compared to $\xi_2(\tau)$, the time dependence of I_1 can be taken into account in a similar fashion to the elimination of the intensity in the phase equation (see equation (14)).

In an initial situation when all the modes are below threshold, we put $I_1 = 0$ in (102). The fluctuations of the mode with largest small-signal gain, I_1 say, are damped least, and it has the highest probability of beginning to oscillate (note that this statement is true only if D is constant in the whole region of interest; as far as spontaneous emission is concerned this occurs in the Doppler limit). Using the terminology of phase transitions mode 1 is a relevant variable (for other analogies see Kazantsev and Surdutovich 1970). The other modes relax rapidly (in times $-1/G_2$) to their equilibrium values, which are modulated by the slowly varying mode. A large fluctuation in I_1 drives mode 1 into the region of positive gain and I_1 rapidly reaches the value corresponding to the oscillating operating point. The increased intensity of the relevant mode further suppresses the small-signal gains of the other, non-oscillating, modes below threshold improving their stability (usually $G(0, 0) < G(I_1, 0)$).

If the escape probabilities of non-oscillating modes are almost equal, whichever mode may begin to oscillate. The stability of single-mode operation is still determined by the small signal gains of the modes that remain non-oscillating. With the aid of the results in § 4.1 we can evaluate their lifetimes, as well as the lifetime of the oscillating mode, and judge whether these are long enough to satisfy the prefixed requirements. When only one mode is at the start above threshold the same considerations hold, except that the mode above threshold will almost certainly be the one that goes over to the oscillating state. For two or more modes initially above threshold the determination of the dynamics becomes very complicated. Even if we completely neglected the noise sources and gave non-vanishing initial values for the modes, we would need semi-classical solutions of general multi-mode operation to evaluate the gain functions of the modes involved in the competition. However, the final state stability considerations are made in a similar way to those of the simpler cases. A more quantitative analysis of the onset of oscillations in a multi-mode laser is outside the scope of this paper.

Acknowledgments

I wish to thank Professor V Ambegaokar and Dr R Graham for useful discussions. I am especially grateful to Dr S Stenholm for many invaluable comments and advice he has given.

Appendix 1

If the field experiences constant damping $G(I) = -G_0$, equation (53) reduces to

$$DIw_n'' + (D + IG_0)w_n' + (E_n + G_0)w_n = 0. \tag{A.1}$$

Introducing the new variable

$$z = \frac{G_0 I}{D}, \tag{A.2}$$

we obtain from (A.1)

$$zw_n'' + (1+z)w_n' + \left(1 + \frac{E_n}{G_0}\right)w_n = 0, \tag{A.3}$$

which has the solution

$$w_n = \mathrm{e}^{-z} L_n(z). \tag{A.4}$$

$L_n(z)$ is the nth order Laguerre polynomial and the nth eigenvalue E_n is

$$E_n = G_0 n \qquad n = 0, 1, 2, \ldots . \tag{A.5}$$

The eigensolutions of (A.1) are thus

$$w_n(I) = \frac{\mathrm{d}z}{\mathrm{d}I} w_n(z) = \frac{G_0}{D} \exp\left(-\frac{G_0 I}{D}\right) L_n\left(\frac{G_0 I}{D}\right). \tag{A.6}$$

Introducing (A.6) into (52) and choosing the initial distribution

$$w(I, 0) = \delta(I - E_0), \tag{A.7}$$

we find

$$\mathscr{G}(I, \tau; E_0, 0) = \frac{G_0}{D} \exp\left(-\frac{G_0 I}{D}\right) \sum_{n=0}^{\infty} \exp(-nG_0\tau) L_n\left(\frac{G_0 I}{D}\right) L_n\left(\frac{G_0 E_0}{D}\right) \tag{A.8}$$

(note that the solution of (52) with the special initial condition (A.7) yields the Green function, this fact is emphasized by writing $\mathscr{G}$ instead of w). Provided that we have $G_0 > 0$, the sum in (A.8) can be expressed in closed form (Bateman 1953, p 188), and we get

$$\mathscr{G}(I, \tau; E_0, 0) = \frac{G_0}{D(1 - \mathrm{e}^{-G_0\tau})} \exp\left(-\frac{G_0(I + E_0 \mathrm{e}^{-G_0\tau})}{D(1 - \mathrm{e}^{-G_0\tau})}\right) I_0\left(\frac{2G_0(IE_0 \mathrm{e}^{-G_0\tau})^{1/2}}{D(1 - \mathrm{e}^{-G_0\tau})}\right), \tag{A.9}$$

where $I_0(x)$ is the modified Bessel function. In the limit $\tau \to \infty$ the function $I_0 \to 1$, and we obtain the correct stationary distribution. For small τ and $E_0 > 0$ the asymptotic expansion of $I_0(x)$ yields

$$\mathscr{G}(I, \tau; E_0, 0) \simeq (4\pi D\tau\sqrt{IE_0})^{-1/2} \exp\left(-\frac{(\sqrt{I} - \sqrt{E_0})^2}{D\tau}\right), \qquad G_0\tau \ll 1, \tag{A.10}$$

which can be shown to satisfy the initial condition (A.7).

In the vicinity of the unstable operating point I_1 we can linearize the drift coefficient $G(I)I$ and neglect the intensity dependence of the diffusion term. The Langevin equation (9) is then

$$\frac{d}{d\tau}(I-I_1) = g_1(I-I_1)+\sqrt{2DI_1}\xi(\tau), \tag{A.11}$$

$$g_1 = G'(I_1)I_1, \tag{A.12}$$

(the assumedly small term $\frac{1}{2}D$ in equation (9) shifts the zero of the driving term by a small amount, but this shift will be neglected). A formal solution of (A.11) satisfying the initial value $I(0) = E_0$ is given by

$$I(\tau)-I_1-(E_0-I_1)\,e^{g_1\tau} = \sqrt{2DI_1}\int_0^\tau d\tau'\,\exp[g_1(\tau-\tau')]\xi(\tau'). \tag{A.13}$$

According to Chandrasekhar (1943) the probability distribution of $I(\tau)$ is

$$\begin{aligned}\mathscr{G}(I,\tau\,;E_0,0) &= \left(\frac{2\pi DI_1}{g_1}(e^{2g_1\tau}-1)\right)^{-1/2}\exp\Bigg[-[I-I_1-(E_0-I_1)\,e^{g_1\tau}]^2\\ &\quad\times\left(\frac{2DI_1}{g_1}(e^{2g_1\tau}-1)\right)^{-1}\Bigg],\end{aligned} \tag{A.14}$$

provided that we have

$$\int_{-\infty}^{0} dI'\mathscr{G}(I',\tau\,;E_0,0) \ll 1. \tag{A.15}$$

The evidently unstable distribution (A.14) is applicable only for times $g_1\tau \ll 1$.

A linearized solution near the stable operating point $I = I_2$ is obtained similarly to (A.11)–(A.14). The result is given by (A.14), if we replace I_1 by I_2 in (A.12) and (A.14):

$$\begin{aligned}\mathscr{G}(I,\tau\,;E_0,0) &= \left(\frac{2\pi DI_2}{g_2}(1-e^{-2g_2\tau})\right)^{-1/2}\exp\Bigg[-[I-I_2-(E_0-I_2)\,e^{-g_2\tau}]^2\\ &\quad\times\left(\frac{2DI_2}{g_2}(1-e^{-2g_2\tau})\right)^{-1}\Bigg],\end{aligned} \tag{A.16}$$

where we have explicitly taken into account the sign of g_2, ie,

$$g_2 = -G'(I_2)I_2. \tag{A.17}$$

The applicability of (A.16) again requires that (A.15) is satisfied.

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